SEQUENCES

EXAMPLE:

1. $a_n = \dfrac{5^{n-1}}{3^n}$, $n \geq 1$: Since we are given $n \geq 1$, the first four terms of the sequence are a_1, a_2, a_3 and a_4. To find a_1 , we replace every occurrence of n in the formula for a_n with $n = 1$ to get $a_1 = \frac{5^{1-1}}{3^1} = \frac{1}{3}$. Proceeding similarly, we get $a_2 = \frac{5^{2-1}}{3^2} = \frac{5}{9}$, $a_3 = \frac{5^{3-1}}{3^3} = \frac{25}{27}$ and $a_4 = \frac{5^{4-1}}{3^4} = \frac{125}{81}$.

$$\frac{1}{3}, \frac{5}{9}, \frac{25}{27}, \frac{125}{81}, \ldots$$

2. $b_k = \dfrac{(-1)^k}{2k+1}$, $k \geq 0$: For this sequence we have $k \geq 0$, so the first four terms are b_0, b_1, b_2 and b_3. Proceeding as before, replacing in this case the variable k with the appropriate whole number, beginning with 0, we get $b_0 = \frac{(-1)^0}{2(0)+1} = 1$, $b_1 = \frac{(-1)^1}{2(1)+1} = -\frac{1}{3}$, $b_2 = \frac{(-1)^2}{2(2)+1} = \frac{1}{5}$ and $b_3 = \frac{(-1)^3}{2(3)+1} = -\frac{1}{7}$. As a side-note, this sequence is called an **alternating sequence** since the signs alternate between '$+$' and '$-$.'

$$1, -\frac{1}{3}, \frac{1}{5}, -\frac{1}{7}, \ldots$$

3. $\{2n-1\}_{n=1}^{\infty}$: The notation $\{2n-1\}_{n=1}^{\infty}$ means $a_n = 2n-1$, $n \geq 1$. We get $a_1 = 1$, $a_2 = 3$, $a_3 = 5$ and $a_4 = 7$.

$$1, 3, 5, 7, \ldots$$

4. $\left\{\dfrac{1+(-1)^i}{i}\right\}_{i=2}^{\infty}$: Here, we are using i as a counter, not as the imaginary unit we saw in Chapter 2. Proceeding as before, we set $a_i = \frac{1+(-1)^i}{i}$, $i \geq 2$. We find $a_2 = 1$, $a_3 = 0$, $a_4 = \frac{1}{2}$ and $a_5 = 0$.

$$1, 0, \frac{1}{2}, 0, \ldots$$

EXAMPLE:

5. $a_1 = 7$, $a_{n+1} = 2 - a_n$, $n \geq 1$:

   To obtain the terms of this sequence, we start with $a_1 = 7$ and use the equation $a_{n+1} = 2 - a_n$ for $n \geq 1$ to generate successive terms. When $n = 1$, this equation becomes $a_{1+1} = 2 - a_1$ which simplifies to $a_2 = 2 - a_1 = 2 - 7 = -5$. When $n = 2$, the equation becomes $a_{2+1} = 2 - a_2$ so we get $a_3 = 2 - a_2 = 2 - (-5) = 7$. Finally, when $n = 3$, we get $a_{3+1} = 2 - a_3$ so $a_4 = 2 - a_3 = 2 - 7 = -5$.

$$7, -5, 7, -5, \ldots$$

6. $f_0 = 1$, $f_n = n \cdot f_{n-1}$, $n \geq 1$:

   As with the problem above, we are given a place to start with $f_0 = 1$ and given a formula to build other terms of the sequence. Substituting $n = 1$ into the equation $f_n = n \cdot f_{n-1}$, we get $f_1 = 1 \cdot f_0 = 1 \cdot 1 = 1$. Advancing to $n = 2$, we get $f_2 = 2 \cdot f_1 = 2 \cdot 1 = 2$. Finally, $f_3 = 3 \cdot f_2 = 3 \cdot 2 = 6$.

$$1, 1, 2, 6, \ldots$$

EXAMPLE: A good rule of thumb here is: 'when in doubt, write it out!' so we write out the first few terms of each sequence and see where that leads us.

7. Writing out the first few terms gives us: $\frac{1}{3}$, $\frac{5}{9}$, $\frac{25}{27}$ and $\frac{125}{81}$. To see if this is an arithmetic sequence, we look at the successive differences of terms. We find that $a_2 - a_1 = \frac{5}{9} - \frac{1}{3} = \frac{2}{9}$ and $a_3 - a_2 = \frac{25}{27} - \frac{5}{9} = \frac{10}{27}$. Since we get different numbers, there is no 'common difference' so the sequence isn't arithmetic.

   To see if the sequence is geometric, we compute the ratios of successive terms. The first three ratios **suggest** the sequence is geometric:

$$\frac{a_2}{a_1} = \frac{\frac{5}{9}}{\frac{1}{3}} = \frac{5}{3}, \quad \frac{a_3}{a_2} = \frac{\frac{25}{27}}{\frac{5}{9}} = \frac{5}{3} \quad \text{and} \quad \frac{a_4}{a_3} = \frac{\frac{125}{81}}{\frac{25}{27}} = \frac{5}{3}$$

   To **prove** the sequence is geometric, however, we must show that $\frac{a_{n+1}}{a_n} = r$ for all n:

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{5^{(n+1)-1}}{3^{n+1}}}{\dfrac{5^{n-1}}{3^n}} = \frac{5^n}{3^{n+1}} \cdot \frac{3^n}{5^{n-1}} = \frac{5}{3}$$

   Hence, the sequence is geometric with common ratio $r = \frac{5}{3}$.

8. Writing out the first four terms of this sequence: 1, $-\frac{1}{3}$, $\frac{1}{5}$ and $-\frac{1}{7}$. We find $b_1 - b_0 = -\frac{4}{3}$ and $b_2 - b_1 = \frac{8}{15}$. Since we get different values, the sequence isn't arithmetic. To see if it is geometric, we compute $\frac{b_1}{b_0} = -\frac{1}{3}$ and $\frac{b_2}{b_1} = -\frac{3}{5}$. Since we get different values, the sequence isn't geometric, either.

9. The sequence $\{2n-1\}_{n=1}^{\infty}$ generates the odd numbers: $1, 3, 5, 7, \ldots$. Computing the first few differences, we find $a_2 - a_1 = 2$, $a_3 - a_2 = 2$, and $a_4 - a_3 = 2$. This suggests that the sequence is arithmetic. To prove this is the case, we find

$$a_{n+1} - a_n = (2(n+1) - 1) - (2n - 1) = 2n + 2 - 1 - 2n + 1 = 2$$

   This establishes that the sequence is arithmetic with common difference $d = 2$. To see if it is geometric, we compute $\frac{a_2}{a_1} = 3$ and $\frac{a_3}{a_2} = \frac{5}{3}$. Since these ratios are different, we conclude the sequence is not geometric.

10. We met our last sequence at the beginning of the section. Given that $a_2 - a_1 = -\frac{5}{4}$ and $a_3 - a_2 = \frac{15}{8}$, the sequence is not arithmetic. Computing the first few ratios, however, gives us $\frac{a_2}{a_1} = -\frac{3}{2}$, $\frac{a_3}{a_2} = -\frac{3}{2}$ and $\frac{a_4}{a_3} = -\frac{3}{2}$. Since these are the only terms given to us, we **assume** that the pattern of ratios continue in this fashion and conclude that the sequence is geometric.

EXAMPLE:

11. This sequence is geometric with common ratio $r = 0.1$. With $a = 0.9$, we get $a_n = 0.9\,(0.1)^{n-1}$ for $n \geq 1$.

12. As the reader can verify, this sequence is neither arithmetic nor geometric. In an attempt to find a pattern, we rewrite the second term with a denominator to make all the terms appear as fractions and associate the '$-$' with the denominators so we have a constant numerator:

$$\frac{2}{5}, \frac{2}{1}, \frac{2}{-3}, -\frac{2}{-7}, \ldots$$

The sequence of the denominators: $5, 1, -3, -7, \ldots$ is arithmetic with a common difference of -4. Plugging in $a = 5$ and $d = -4$, we get a formula for the nth denominator as $5 + (n-1)(-4) = 9 - 4n$ for $n \geq 1$. Hence, our final answer is $a_n = \frac{2}{9-4n}$, $n \geq 1$.

13. The sequence as given is neither arithmetic nor geometric, so we proceed as in the last problem to try to get patterns individually for the numerator and denominator. Letting C_n and D_n denote the sequence of numerators and denominators, respectively, so that $a_n = \frac{C_n}{D_n}$.

After some experimentation, we choose to write the first term as a fraction and associate the negatives '$-$' with the numerators. This yields

$$\frac{1}{1}, \frac{-2}{7}, \frac{4}{13}, \frac{-8}{19}, \ldots$$

The numerators form the sequence $1, -2, 4, -8, \ldots$ which is geometric with $a = 1$ and $r = -2$, so we get $C_n = (-2)^{n-1}$, for $n \geq 1$.

The denominators $1, 7, 13, 19, \ldots$ form an arithmetic sequence with $a = 1$ and $d = 6$. Hence, we get $D_n = 1 + 6(n-1) = 6n - 5$, for $n \geq 1$.

Putting these two formulas together, we obtain our formula for $a_n = \frac{C_n}{D_n} = \frac{(-2)^{n-1}}{6n-5}$, for $n \geq 1$. We leave it to the reader to show that this checks out.